Calculus II - Day 13

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Goals for today

- Compute both definite and indefinite integrals of the form $\int u(x)v'(x)\,dx$.
- Decide which function is best to choose for $u(x)$.
- Solve "wrap-around" IBP (Integration by Parts) problems.

Recall:

The Product Rule:
$$\frac{d}{dx}(f(x)\cdot g(x)) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

Let's integrate both sides:

$$\int \frac{d}{dx}(f(x)g(x))\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

Rearrange:
$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Challenge: Recognize $h(x) = f(x)\cdot g'(x)$.
Another way of writing this: If $u = u(x)$ and $v = v(x)$ are differentiable functions of x, then

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example: $\int x \sin(x)\, dx$

Idea: Write $x \sin(x)\, dx$ as udv where - u: "easy to differentiate" - dv: "easy to integrate" (dx always goes here)

$u = x, \quad du = dx$
$dv = \sin(x)\, dx, \quad v = -\cos(x)$

$$uv - \int v\, du = -x \cos(x) + \int \cos(x)\, dx$$
$$= -x \cos(x) + \sin(x) + C$$

Example:

$$\int x^2 e^x\, dx$$

Choose $u = x^2$ and $dv = e^x\, dx$:

$$u = x^2, \quad du = 2x\, dx$$

$$dv = e^x\, dx, \quad v = e^x$$

Applying integration by parts:

$$uv - \int v\, du = x^2 e^x - \int e^x \cdot 2x\, dx$$

Now, we need to apply integration by parts again to $\int 2xe^x\, dx$:
Choose $u = 2x$ and $dv = e^x\, dx$:

$$u = 2x, \quad du = 2\, dx$$
$$dv = e^x\, dx, \quad v = e^x$$

Then,

$$uv - \int v\, du = 2xe^x - \int e^x \cdot 2\, dx = 2xe^x - 2e^x + C$$

Returning to the original integral:

$$\int x^2 e^x\, dx = x^2 e^x - \int e^x \cdot 2x\, dx$$
$$= x^2 e^x - (2xe^x - 2e^x + C)$$
$$= x^2 e^x - 2xe^x + 2e^x + C$$

Question: How do you choose u and dv?

u: should be "simpler" after taking the derivative.
dv: shouldn't be too difficult to integrate.

Acronym: LIPPET

better choice for u ↑	**L**	log functions
	I	inverse trig functions
	P	polynomials
	P	powers of x
	E	exponentials e^x
worse choice for u / better choice for dv ↓	**T**	trig functions

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Example: You try: $\int x^4 \ln(x)\, dx$
Choose $u = \ln(x)$ and $dv = x^4\, dx$:

$$u = \ln(x), \quad du = \frac{1}{x}\, dx$$

$$dv = x^4\, dx, \quad v = \frac{1}{5}x^5$$

Applying integration by parts:

$$\begin{aligned} uv - \int v\, du &= \frac{1}{5}x^5 \ln(x) - \int \frac{1}{5}x^5 \cdot \frac{1}{x}\, dx \\ &= \frac{1}{5}x^5 \ln(x) - \frac{1}{5}\int x^4\, dx \\ &= \frac{1}{5}x^5 \ln(x) - \frac{1}{5} \cdot \frac{1}{5}x^5 + C \\ &= \frac{1}{5}x^5 \ln(x) - \frac{1}{25}x^5 + C \end{aligned}$$

So, the final answer is:

$$\int x^4 \ln(x)\, dx = \frac{1}{5}x^5 \ln(x) - \frac{1}{25}x^5 + C$$

Another Example: $\int \ln(x)\, dx$
Choose $u = \ln(x)$ and $dv = dx$:

$$u = \ln(x), \quad du = \frac{1}{x}\, dx$$

$$dv = dx, \quad v = x$$

Applying integration by parts:

$$uv - \int v\, du = x \ln(x) - \int x \cdot \frac{1}{x}\, dx$$

$$= x\ln(x) - \int 1\,dx$$
$$= x\ln(x) - x + C$$

So, the final answer is:

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

Example: "Wrap-around" IBP

$$\int e^x \sin(x)\,dx$$

Choose $u = e^x$ and $dv = \sin(x)\,dx$:

$$u = e^x, \quad du = e^x\,dx$$
$$dv = \sin(x)\,dx, \quad v = -\cos(x)$$

Applying integration by parts:

$$uv - \int v\,du = -e^x\cos(x) + \int e^x\cos(x)\,dx$$

Now we apply integration by parts again to $\int e^x\cos(x)\,dx$.

$$= -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\,dx$$

Notice that we have the original integral $\int e^x\sin(x)\,dx$ on both sides:

$$\int e^x\sin(x)\,dx = -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\,dx$$

Add $\int e^x\sin(x)\,dx$ to both sides:

$$2\int e^x\sin(x)\,dx = -e^x\cos(x) + e^x\sin(x)$$
$$\int e^x\sin(x)\,dx = \frac{1}{2}\left(-e^x\cos(x) + e^x\sin(x)\right) + C$$

Transition to Definite Integrals:
For integration by parts with definite integrals:

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,dx$$

Or in alternative notation:

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

Example: $\int_0^{\pi/2} x^2 \cos(x)\,dx$
Choose $u = x^2$ and $dv = \cos(x)\,dx$:

$$u = x^2, \quad du = 2x\,dx$$

$$dv = \cos(x)\,dx, \quad v = \sin(x)$$

Applying integration by parts:

$$\int_0^{\pi/2} x^2 \cos(x)\,dx = \left[x^2 \sin(x)\right]_0^{\pi/2} - \int_0^{\pi/2} \sin(x) \cdot 2x\,dx$$

Now, apply integration by parts again to $\int_0^{\pi/2} 2x \sin(x)\,dx$.
Choose $u = 2x$ and $dv = \sin(x)\,dx$:

$$u = 2x, \quad du = 2\,dx$$

$$dv = \sin(x)\,dx, \quad v = -\cos(x)$$

Then,

$$\int_0^{\pi/2} x^2 \cos(x)\,dx = \left[x^2 \sin(x)\right]_0^{\pi/2} + [-2x\cos(x)]_0^{\pi/2} + \int_0^{\pi/2} 2\cos(x)\,dx$$

Evaluating each term:

$$= \left[x^2 \sin(x)\right]_0^{\pi/2} + [2x\cos(x)]_0^{\pi/2} - [2\sin(x)]_0^{\pi/2}$$

Substitute the limits for each term:

$$= \left(\frac{\pi^2}{4} \cdot 1 + \pi \cdot 0 - 2 \cdot 1\right) - (0 + 0 - 0)$$

$$= \frac{\pi^2}{4} - 2$$

So, the final answer is:

$$\int_0^{\pi/2} x^2 \cos(x)\,dx = \frac{\pi^2}{4} - 2$$

Example: $\int_0^1 \arctan(x)\,dx$
Choose $u = \arctan(x)$ and $dv = dx$:

$$u = \arctan(x), \quad du = \frac{1}{1+x^2}\,dx$$

$$dv = dx, \quad v = x$$

Applying integration by parts:

$$\int_0^1 \arctan(x)\,dx = [x\arctan(x)]_0^1 - \int_0^1 x \cdot \frac{1}{1+x^2}\,dx$$

Now, apply a u-substitution on $\int_0^1 x \cdot \frac{1}{1+x^2}\,dx$: Let $u = 1 + x^2$, then $du = 2x\,dx$.

$$u(0) = 1 + 0^2 = 1, \quad u(1) = 1 + 1^2 = 2$$

$$\int_0^1 x \cdot \frac{1}{1+x^2}\,dx = \int_1^2 \frac{1}{2} \cdot \frac{1}{u}\,du = \frac{1}{2} \ln|u| \Big|_1^2$$

$$= \frac{1}{2}\ln(2) - \frac{1}{2}\ln(1) = \frac{1}{2}\ln(2)$$

Returning to the original integral:

$$\int_0^1 \arctan(x)\,dx = [x\arctan(x)]_0^1 - \frac{1}{2}\ln(2)$$

$$= \left(\frac{\pi}{4} \cdot 1 - 0\right) - \frac{1}{2}\ln(2)$$

$$= \boxed{\frac{\pi}{4} - \frac{1}{2}\ln(2)}$$

Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

Dividing each term by $\sin^2(x)$:

$$1 + \cot^2(x) = \csc^2(x)$$

Dividing each term by $\cos^2(x)$:

$$\tan^2(x) + 1 = \sec^2(x)$$

Half-Angle Formulas

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

Goal: Integrate functions of the form

$$\int \sin^m(x)\cos^n(x)\,dx$$